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Chains of Darboux transformations for the matrix Schrödinger equation

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Abstract
Chains of Darboux transformations for the matrix Schrödinger equation are considered. A matrix generalization of the well-known for the scalar equation Crum–Krein formulae for the resulting action of such chains is given.

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1. Introduction

Let us consider the matrix Schrödinger equation

$$h_0\Psi_E = E\Psi_E \qquad h_0 = -D^2 + V_0(x) \qquad D \equiv \frac{\mathrm{d}}{\mathrm{d}x} \tag{1}$$

where $V_0(x)$ is an $n \times n$ Hermitian matrix with x-dependent entries, $\Psi_E = (\psi_{1E}, \ldots, \psi_{nE})^t$ is a vector of an n-dimensional linear space, and E is a number which plays an essential role in different physical applications. For instance, a multichannel quantum system may be described by this equation [1]. One of the most interesting applications of this equation consists in the possibility of involving the supersymmetric quantum mechanics [2] for describing the scattering of composite particles such as atom–atom or nucleon–nucleon collisions [3–5]. In particular, in this way one can interpret an ambiguity between shallow and deep potentials of the nucleon–nucleon interaction [5]. To get a qualitative result, supersymmetric transformations were successively applied in [3–5], which required performing a lot of unnecessary work. In this way one can realize only a few transformation steps. We believe that progress in applications of this method is essentially delayed because of the absence of a simple possibility of getting rid of intermediate Hamiltonians and going directly to the final result of a chain of transformations. We notice that such a possibility exists for the usual (scalar) Schrödinger equation, being given by the known Crum–Krein determinant formulae [6, 7] which made it possible to get a number of new interesting applications of one channel supersymmetric

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quantum mechanics in describing the nucleon–nucleon scattering [8]. Nevertheless, this problem was tackled by Goncharenko and Veselov [9] when they realized that Gelfand–Retakh quasideterminants [10] may be used for this purpose. We would like to point out that although their method gives a solution in principle, this is very complicated and difficult for practical realization since it involves a matrix calculus with matrices defined over a noncommutative ring and in particular it is necessary to invert such matrices.

In this paper we prove alternative formulae where only the usual determinants are involved. They are very similar to the known Crum–Krein determinant formulae and can be considered as their straightforward generalizations.

2. First-order transformation

We follow the definition of the Darboux transformation operator given by Goncharenko and Veselov [9], defining it as a first-order differential operator with matrix-valued coefficients

$$L = L_0(x) + L_1(x)D \tag{2}$$

intertwining h_0 and h_1

$$Lh_0 = h_1 L \tag{3}$$

where h_0 is introduced above and h_1 is defined by the potential V_1,

$$h_1 = -D^2 + V_1(x). \tag{4}$$

If such an operator is found one can get eigenfunctions of h_1, by the simple action of L on the eigenfunctions of h_0,

$$\Phi_E = L\Psi_E = (\varphi_{1E}, \ldots, \varphi_{nE})^t \qquad h_1\Phi_E = E\Phi_E. \tag{5}$$

Since equations (1) and (5) are homogeneous, without loss of generality we can put L_1 equal to the identity, $L_1 = 1$ (we suppose $\det L_1 \neq 0$). After inserting L into the intertwining relation (3) we get a system of equations for L_0 and the transformed potential V_1. It is not difficult to find the solution to this system [9]. Thus, L is given by

$$L = D - F \qquad F = \mathcal{U}'\mathcal{U}^{-1} \tag{6}$$

and for the potential V_1 one obtains

$$V_1 = V_0 - 2F'. \tag{7}$$

The matrix-valued function $\mathcal{U}$ is a solution to the equation

$$h_0\mathcal{U} = \mathcal{U}\Lambda \tag{8}$$

where Λ is a constant matrix.

The known supersymmetric approach [2–5] is based on the factorization of the Hamiltonian

$$h_0 = L^+L + \lambda I \qquad \lambda \in \mathbb{R} \tag{9}$$

where L^+ is defined with the help of the formal relations: $D^+ = -D$, $i^+ = -i$, $(AB)^+ = B^+A^+$ and I is the identity matrix. To compare our method with this technique let us consider the superposition of L and its conjugate. After a simple algebra one finds

$$L^+L = h_0 - \mathcal{U}\Lambda\mathcal{U}^{-1}. \tag{10}$$

In particular, when $\Lambda = \lambda I$, we recover the factorization (9) giving rise to the supersymmetry with λ meaning the factorization constant. Similarly, the inverse superposition reads

$$LL^+ = h_1 - \mathcal{U}\Lambda\mathcal{U}^{-1}. \tag{11}$$

If Λ is a diagonal matrix $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ then the system of equations (8) just takes the form of the Schrödinger equation for the columns $U_j = (u_{j,1}, \ldots, u_{j,n})^t$ of the matrix $\mathcal{U} = (U_1, \ldots, U_n)$

$$h_0 U_j = \lambda_j U_j \qquad j = 1, \ldots, n. \tag{12}$$

This means that if we know solutions of the Schrödinger equation (1) then solutions of equation (8) are also known for the diagonal form of the eigenvalue matrix Λ. Therefore in what follows we will consider only diagonal Λ.

3. Chains of Darboux transformations

Now we want to consider chains of transformations defined in the previous section. Chains appear naturally if we note that if sufficiently many matrix solutions to the initial equation are known then any such solution is transformed into a matrix solution of the new equation and, hence, the latter may play the role of the initial equation for the next transformation step.

Suppose we know N matrix solutions of equation (8) corresponding to different eigenvalue matrices $\Lambda_k \neq \Lambda_l$,

$$h_0 \mathcal{U}_k = \mathcal{U}_k \Lambda_k \qquad k = 1, \ldots, N. \tag{13}$$

For the first transformation step we take the function $\mathcal{U}_1$ and according to (6) construct the transformation operator

$$L_{1\leftarrow 0} = D - \mathcal{U}_1' \mathcal{U}_1^{-1}. \tag{14}$$

We note that it can be applied not only on vector-valued functions such as Ψ_E but also on matrix-valued functions, $\mathcal{U}_2, \ldots, \mathcal{U}_N$. In this way we get the matrix solutions $\mathcal{V}_2 = L_{1\leftarrow 0}\mathcal{U}_2, \ldots, \mathcal{V}_N = L_{1\leftarrow 0}\mathcal{U}_N$ of the equation with the potential

$$V_1 = V_0 - 2F_1' \qquad F_1 = \mathcal{U}_1' \mathcal{U}_1^{-1}. \tag{15}$$

Now $\mathcal{V}_2$ can be taken as the transformation function for the Hamiltonian $h_1 = -D^2 + V_1$ to produce the potential

$$V_2 = V_1 - 2\big(\mathcal{V}_2' \mathcal{V}_2^{-1}\big)' = V_0 - 2F_2' \qquad F_2 = F_1 + \mathcal{V}_2' \mathcal{V}_2^{-1} \tag{16}$$

and the transformation operator $L_{2\leftarrow 1} = D - \mathcal{V}_2' \mathcal{V}_2^{-1}$ and so on, till one obtains the potential

$$V_N = V_0 - 2F_N' \tag{17}$$

with F_N defined recursively

$$F_N = F_{N-1} + Y_N' Y_N^{-1} \qquad N = 1, 2, \ldots \qquad F_0 = 0 \tag{18}$$

and Y_N being the matrix-valued solution at the $(N-1)$th step of transformation

$$Y_N = L_{(N-1)\leftarrow(N-2)} \ldots L_{2\leftarrow 1} L_{1\leftarrow 0} \mathcal{U}_N \equiv L_{(N-1)\leftarrow 0} \mathcal{U}_N \tag{19}$$

which produces the final transformation operator $L_{N\leftarrow(N-1)} = -D + Y_N' Y_N^{-1}$.

To get the final potential V_N one has to calculate all intermediate transformation functions performing substantial numerical work even for the scalar case. In practical calculations one can perform only a few steps which restrict considerably possible applications of the method. Fortunately, for the scalar case there exist what are called Crum [6] or Crum–Krein [7] determinant formulae which allow one to omit all intermediate steps and go directly from h_0 to h_N. The function

$$\varphi_E = \frac{|W(u_1, \ldots, u_N, \psi_E)|}{|W(u_1, \ldots, u_N)|} \tag{20}$$

is an eigenfunction of the Hamiltonian h_N with the potential

$$V_N = V_0 - 2\left(\frac{|\tilde{W}(u_1,\ldots,u_N)|}{|W(u_1,\ldots,u_N)|}\right)' \tag{21}$$

provided all $u_k, k = 1,\ldots,N$ and ψ_E are eigenfunctions of the initial Hamiltonian h_0 with the scalar potential V_0: $h_0 = -D^2 + V_0, h_0u_k = \alpha_k u_k, h_0\psi_E = E\psi_E$. Here and in what follows the symbol $|\cdot|$ means the usual determinant, $W(u_1,\ldots,u_N)$ is the Wronsky matrix

$$W(u_1,\ldots,u_N) = \begin{pmatrix} u_1 & u_2 & \ldots & u_N \\ u_1' & u_2' & \ldots & u_N' \\ \ldots & \ldots & \ldots & \ldots \\ u_1^{(N-1)} & u_2^{(N-1)} & \ldots & u_N^{(N-1)} \end{pmatrix} \tag{22}$$

and the matrix $\tilde{W}(u_1,\ldots,u_N)$ is obtained from $W(u_1,\ldots,u_N)$ by replacing its last row composed of $u_k^{(N-1)}$ with $u_k^{(N)}$, $k = 1,\ldots,N$. Of course, the determinant $|\tilde{W}(u_1,\ldots,u_N)|$ is nothing but the derivative of the determinant of the Wronsky matrix $|W(u_1,\ldots,u_N)|$, but we write the second logarithmic derivative of the Wronskian $|W(u_1,\ldots,u_N)|$ in (21) as the first derivative of the ratio of corresponding determinants to stress the similarity of this scalar formula and its matrix generalization below. Formula (20) defines for the scalar case the superposition of the operators of type (14) with the replacement of the matrix-valued functions $\mathcal{U}_k$ by the usual functions u_k,

$$L_{N\leftarrow 0} = L_{N\leftarrow(N-1)}\ldots L_{2\leftarrow 1}L_{1\leftarrow 0} \tag{23}$$

$$\varphi_E = L_{N\leftarrow 0}\psi_E. \tag{24}$$

We shall prove below generalizations of the formulae (20) and (21) to the matrix case meaning that we shall solve the recursion defined by (18), (19) and find the superposition of the operators (23), but first we need to introduce some new notation and to prove an additional statement.

3.1. Notation

Define first the nN-dimensional square Wronsky matrix

$$W(\mathcal{U}_1,\ldots,\mathcal{U}_N) = \begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 & \cdots & \mathcal{U}_N \\ \mathcal{U}_1' & \mathcal{U}_2' & \cdots & \mathcal{U}_N' \\ \ldots & \ldots & \ldots & \ldots \\ \mathcal{U}_1^{(N-1)} & \mathcal{U}_2^{(N-1)} & \cdots & \mathcal{U}_N^{(N-1)} \end{pmatrix}. \tag{25}$$

Here $\mathcal{U}_k$ are $n\times n$ matrices

$$\mathcal{U}_k = \begin{pmatrix} u_{1,1;k} & u_{1,2;k} & \cdots & u_{1,n;k} \\ u_{2,1;k} & u_{2,2;k} & \cdots & u_{2,n;k} \\ \ldots & \ldots & \ldots & \ldots \\ u_{n,1;k} & u_{n,2;k} & \cdots & u_{n,n;k} \end{pmatrix} \qquad k = 1,\ldots,N \tag{26}$$

with columns being n-dimensional vectors $U_{j;k} = (u_{1,j;k},\ldots,u_{n,j;k})^t$ so that $\mathcal{U}_k = (U_{1;k},\ldots,U_{n;k}), k = 1,\ldots,N$. We can also present $\mathcal{U}_k$ as a collection of rows $U_k^j = (u_{j,1;k},\ldots,u_{j,n;k}), j = 1,\ldots,n, \mathcal{U}_k = \left(U_k^1,\ldots,U_k^n\right)^t$. It will be convenient to present (25) also in the form

$$W(\mathcal{U}_1,\ldots,\mathcal{U}_N) = \begin{pmatrix} & & & \mathcal{U}_N \\ & W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1}) & & \mathcal{U}_N' \\ & & & \ldots \\ \mathcal{U}_1^{(N-1)} & \ldots & \mathcal{U}_{N-1}^{(N-1)} & \mathcal{U}_N^{(N-1)} \end{pmatrix} \tag{27}$$

stressing its recursion nature.

Introduce also the matrix

$$W_{jE}(\mathcal{U}_1,\ldots,\mathcal{U}_N)=\begin{pmatrix} & & & \Psi_E \\ & W(\mathcal{U}_1,\ldots,\mathcal{U}_N) & & \Psi_E' \\ & & & \cdots \\ & & & \Psi_E^{(N-1)} \\ \left(U_1^j\right)^{(N)} & \cdots & \left(U_N^j\right)^{(N)} & \psi_{jE}^{(N)} \end{pmatrix} \tag{28}$$

recalling that U_k^j is the jth row of the matrix $\mathcal{U}_k$.

We shall also need the following matrix:

$$W_j^i(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})=\begin{pmatrix} & & & U_{i;N} \\ & W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1}) & & U_{i;N}' \\ & & & \cdots \\ & & & U_{i;N}^{(N-2)} \\ \left(U_1^j\right)^{(N-1)} & \cdots & \left(U_{N-1}^j\right)^{(N-1)} & u_{j,i;N}^{(N-1)} \end{pmatrix}. \tag{29}$$

First we note that this is the previous matrix where N is replaced with $N-1$ and Ψ_E with the vector $U_{i;N}$. Another useful remark is that the determinants $\left|W_j^i(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})\right|$, $i,j=1,\ldots,n$ are nothing but minors embordering the block $W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})$ in the determinant of the matrix (27). (For the definition of embordering minors see the appendix.)

Finally we introduce the matrices $W_{i,j}(\mathcal{U}_1,\ldots,\mathcal{U}_N)$, $i,j=1,\ldots,n$, constructed from the Wronsky matrix (25) where the last matrix row composed of matrices $\mathcal{U}_k^{(N-1)}$ is replaced by $\mathcal{U}_k^{ij}$, $k=1,\ldots,N$:

$$W_{i,j}(\mathcal{U}_1,\ldots,\mathcal{U}_N)=\begin{pmatrix} \mathcal{U}_1 & \mathcal{U}_2 & \cdots & \mathcal{U}_N \\ \mathcal{U}_1' & \mathcal{U}_2' & \cdots & \mathcal{U}_N' \\ \cdots & \cdots & \cdots & \cdots \\ \mathcal{U}_1^{(N-2)} & \mathcal{U}_2^{(N-2)} & \cdots & \mathcal{U}_N^{(N-2)} \\ \mathcal{U}_1^{ij} & \mathcal{U}_2^{ij} & \cdots & \mathcal{U}_N^{ij} \end{pmatrix}. \tag{30}$$

The matrices $\mathcal{U}_k^{ij}$, $i,j=1,\ldots,n$ are constructed from the matrix $\mathcal{U}_k^{(N-1)}$ by replacing its jth row with the ith row of the matrix $\mathcal{U}_k^{(N)}$.

3.2. Main lemma

In this subsection we prove a lemma we are using in the proof of theorems below. Moreover, in proving it as well as the theorems we are using the *Sylvester identity* [11] which is formulated in the appendix.

Consider the matrix

$$A=\begin{pmatrix} a_{1,1} & \cdots & a_{1,p} & a_{1,p+1} & \cdots & a_{1,p+n} \\ & \cdots & & & \cdots & \\ a_{p,1} & \cdots & a_{p,p} & a_{p,p+1} & \cdots & a_{p,p+n} \\ b_{1,1} & \cdots & b_{1,p} & b_{1,p+1} & \cdots & b_{1,p+n} \\ b_{2,1} & \cdots & b_{2,p} & b_{2,p+1} & \cdots & b_{2,p+n} \end{pmatrix}. \tag{31}$$

Let a be the $p\times p$ submatrix of A with the entries $a_{i,j}$, $i,j=1,\ldots,p$. Denote by m_{jk} the minor of A embordering a with the jth ($j=1,2$) row composed of $b_{j,i}$, $j=1,2$, $i=1,\ldots,p$ and the kth column ($p<k\leqslant p+n$). Let also m_{jk}^{ts} be the minor obtained from m_{jk} by replacing its sth row composed of $a_{s,j}$ ($s\leqslant p$) with the $(p+t)$th row composed of

$b_{t,i}$ $(t = 1, 2)$. Let now a^{ts} be obtained from a with the help of the same replacement, i.e. with the replacement of its sth row composed of $a_{s,j}$ $(s \leqslant p)$ by the $(p + t)$th row of A composed of $b_{t,j}$ $(t = 1, 2)$.

Lemma 1. *If* $|a| \neq 0$ *then we have the following determinant identity*

$$|a|m^{ts}_{jk} = |a^{ts}|m_{jk} - |a^{js}|m_{tk}. \tag{32}$$

Proof. Consider an auxiliary square matrix

$$A_{jt} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,p} & a_{1,k} & 0 \\ & \cdots & & & \cdots \\ a_{s,1} & \cdots & a_{s,p} & a_{s,k} & 0 \\ & \cdots & & & \cdots \\ a_{p,1} & \cdots & a_{p,p} & a_{p,k} & 0 \\ b_{j,1} & \cdots & b_{j,p} & b_{j,k} & 1 \\ b_{t,1} & \cdots & b_{t,p} & b_{t,k} & 1 \end{pmatrix} \tag{33}$$

where $j,\ t = 1, 2$ and the last column contains only two nonzero entries. Take its main minor $|a|$. There are only four minors of A_{jt} embordering $|a|$. Minors m_{jk} and $|a|$ emborder it by the row $b_{j,i},\ i = 1, \ldots, p$ and the next to last and the last columns respectively and minors m_{tk} and $|a|$ emborder it with the last row and the same columns. According to the Sylvester identity the determinant composed of these embordering minors is equal to

$$m_{jk}|a| - m_{tk}|a| = |a||A_{jt}| \tag{34}$$

where we can cancel $|a|$ since it is supposed to be different from zero.

Now interchange in the matrix A_{jt} the sth and the last rows to get

$$\tilde{A}_{jt} = \begin{array}{@{}c@{}l} \begin{pmatrix} a_{11} & \cdots & a_{1p} & a_{1k} & 0 \\ & \cdots & & & \cdots \\ b_{t1} & \cdots & b_{tp} & b_{tk} & 1 \\ & \cdots & & & \cdots \\ a_{p1} & \cdots & a_{pp} & a_{pk} & 0 \\ b_{j1} & \cdots & b_{jp} & b_{jk} & 1 \\ a_{s1} & \cdots & a_{sp} & a_{sk} & 0 \end{pmatrix} & \begin{array}{l} \\ \\ \leftarrow s\text{th row} \\ \\ \\ \\ \\ \end{array} \end{array}. \tag{35}$$

The upper-left block of this matrix of dimension $p \times p$ is the above-introduced matrix a^{ts}. Let us find embordering minors for this submatrix. It is clear that m^{ts}_{jk} and $-m_{tk}$ emborder it with the $(p + 1)$th column and the $(p + 1)$th and the $(p + 2)$th rows respectively. The minor embordering a^{ts} with the $(p + 2)$th column and $(p + 1)$th row has in the last column only two nonzero entries which are equal to one. Therefore we can decompose it on this column and after corresponding interchange of the rows one gets for it the expression $|a^{ts}| - |a^{js}|$. The last minor embordering a^{ts} with the $(p+2)$th column and $(p+2)$th row is equal to $-|a|$ which becomes evident after corresponding interchange of the rows. Once again we consider the determinant composed of these minors and calculate it using the Sylvester identity

$$|\tilde{A}_{jt}||a^{ts}| = m_{tk}|a^{ts}| - m_{tk}|a^{js}| - |a|m^{ts}_{jk}. \tag{36}$$

Since $|\tilde{A}_{jt}| = -|A_{jt}|$ the lemma follows from equations (34) and (36). $\square$

3.3. Transformation of vectors

In this subsection we formulate and prove the theorem about the transformation of a vector Ψ_E by a chain of transformations introduced at the beginning of this section.

Theorem 1. *The resulting action of a chain of Darboux transformations applied to a vector*

$$\Psi_E = (\psi_{1E}, \ldots, \psi_{NE})^t \tag{37}$$

is the vector

$$\Phi_E = L_{N\leftarrow(N-1)} \ldots L_{2\leftarrow 1} L_{1\leftarrow 0} \Psi_E = (\varphi_{1E}, \ldots, \varphi_{NE})^t \tag{38}$$

with the entries φ_{jE} *given by*

$$\varphi_{jE} = \frac{|W_{jE}(\mathcal{U}_1, \ldots, \mathcal{U}_N)|}{|W(\mathcal{U}_1, \ldots, \mathcal{U}_N)|} \tag{39}$$

where $W_{jE}(\mathcal{U}_1, \ldots, \mathcal{U}_N)$ *is given in (28) and* $W(\mathcal{U}_1, \ldots, \mathcal{U}_N)$ *is defined by (25).*

Proof. To prove theorem 1 we are using the perfect induction method. So, first we shall prove it for $N = 1$. In this case

$$\Phi_E = L_{1\leftarrow 0}\Psi_E = \left(D - \mathcal{U}_1'\mathcal{U}_1^{-1}\right)\Psi_E = (\varphi_{1E}, \ldots, \varphi_{nE})^t. \tag{40}$$

Denote by A_{ij} the cofactor of the element $u_{j,i;1}$ in the matrix $\mathcal{U}_1$. Then according to the definition of the inverse matrix one has

$$(\mathcal{U}^{-1}\Psi_E)_j = \frac{1}{|\mathcal{U}_1|}\sum_{i=1}^{n} A_{ij}\psi_{iE} \tag{41}$$

which for the elements of the vector Φ_E implies

$$\varphi_{lE} = \partial\psi_{lE} - \frac{1}{|\mathcal{U}_1|}\sum_{i,j=1}^{n} u'_{l,j;1}A_{ij}\psi_{iE} \equiv \frac{\Delta_{lE}}{|\mathcal{U}_1|}. \tag{42}$$

Consider now the matrix

$$W_{jE}(\mathcal{U}_1) = \begin{pmatrix} & & & \psi_{1E} \\ & \mathcal{U}_1 & & \ldots \\ & & & \psi_{nE} \\ u'_{j,1;1} & \cdots & u'_{j,n;1} & \psi'_{jE} \end{pmatrix}. \tag{43}$$

If we decompose the determinant $|W_{jE}(\mathcal{U}_1)|$ on the elements of the last row and all determinants appearing in this decomposition except for the one coinciding with $|\mathcal{U}_1|$ decompose on the elements of the last column, the resulting expression will coincide exactly with the numerator of the right-hand side of (42) meaning that

$$\varphi_{jE} = \frac{|W_{jE}(\mathcal{U}_1)|}{|\mathcal{U}_1|} \tag{44}$$

which proves the assertion for $N = 1$.

Suppose theorem 1 holds for the chain of $N - 1$ transformations which means that the following vector gives the resulting action of this chain:

$$\Theta_E = L_{(N-1)\leftarrow 0}\Psi_E = (\theta_{1E}, \ldots, \theta_{nE})^t \tag{45}$$

where

$$\theta_{jE} = \frac{|W_{jE}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|}{|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|}. \tag{46}$$

Now since $L_{N\leftarrow 0} = L_{N\leftarrow(N-1)}L_{(N-1)\leftarrow 0}$ we have to apply the first-order operator $L_{N\leftarrow(N-1)}$ to the vector (45) but first we need to act with $L_{(N-1)\leftarrow 0}$ on the vectors $U_{i;N},\ i = 1,\dots,n$ which form the columns of the matrix-valued transformation function $\mathcal{U}_N$ to find the transformation function $Y_N = L_{(N-1)\leftarrow 0}\mathcal{U}_N$ for the Nth transformation step and determine the operator $L_{N\leftarrow(N-1)} = -D + Y_N' Y_N^{-1}$. By supposition of (45) and (46), this result may be rewritten as follows:

$$L_{(N-1)\leftarrow 0}U_{i;N} = (y_{1,i},\dots,y_{n,i})^t \tag{47}$$

$$y_{j,i} = \frac{\left|W_j^i(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})\right|}{|W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})|} \qquad i,j = 1,\dots,n. \tag{48}$$

This means that the matrix Y_N has $y_{j,i}$ (48) as its entry. If we note that they coincide up to the constant factor $1/|W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})|$ with the minors embordering the block $W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})$ in the matrix (27) we can apply the Sylvester identity to calculate the determinant $|Y_N|$:

$$|Y_N| = \frac{|W(\mathcal{U}_1,\dots,\mathcal{U}_N)|}{|W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})|}. \tag{49}$$

We need this determinant since the action of the first-order Darboux transformation operator on a vector is given by (40), (43) and (44) meaning that the vector

$$\Phi_E = \left[D - Y_N' Y_N^{-1}\right]\Theta_E \tag{50}$$

has the entries

$$\varphi_{jE} = \left|Y_{NE}^j\right|/|Y_N| \qquad j = 1,\dots,n \tag{51}$$

with

$$Y_{NE}^j = \begin{pmatrix} & & & \theta_{1E} \\ & Y_N & & \dots \\ & & & \theta_{nE} \\ y_{j,1}' & \cdots & y_{j,n}' & \theta_{jE}' \end{pmatrix}. \tag{52}$$

To calculate the entries of Y_{NE}^j we have to differentiate $y_{j,i}$ (48):

$$y_{j,i}' = -\frac{|W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})|'}{|W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})|}y_{j,i} + \frac{\left|W_j^i(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})\right|'}{|W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})|}. \tag{53}$$

Now we first calculate the derivative of the determinant $\left|W_j^i\right|$ keeping in this expression the derivative of its last row as a separate term

$$\left|W_j^i(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})\right|' = \Gamma_{ji} + \sum_{m=1(\neq j)}^{n} \Delta_{ji}^m. \tag{54}$$

Here Γ_{ji} is the determinant of the same matrix (29) where only the last row is differentiated and Δ_{ji}^m is the determinant of the same matrix where in the last matrix row of the block $W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})$ one has to differentiate only the mth row of the matrices $\mathcal{U}_k^{(N-2)},\ k = 1,\dots,N$.

We note that the matrix of the determinant Δ_{ji}^m can also be obtained by interchanging one of the rows of matrices $\mathcal{U}_k^{(N-2)}$ with a row of matrices $\mathcal{U}_k^{(N-1)},\ k = 1,\dots,N$ in the minor $\left|W_j^i(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})\right|$ when it is considered as a minor embordering the block $W(\mathcal{U}_1,\dots,\mathcal{U}_{N-1})$ in the matrix (27). This matrix contains two rows with the derivatives of the $(N-1)$th order of elements of matrices $\mathcal{U}_k$ in contrast with any minor embordering the block $W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})$ in the matrix (27) which contains only one such row. Therefore there is no way to apply the Sylvester identity for calculating this determinant. Just for this purpose we have proved our main lemma which yields

$$|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|\Delta_{ji}^m = |W_{mm}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|\left|W_j^i(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})\right| - |W_{jm}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|\left|W_m^i(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})\right|. \tag{55}$$

Now from (54) and (55) one obtains

$$\left|W_j^i(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})\right|' = \frac{1}{|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|}\Bigg\{\Gamma_{ji}|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})| + \sum_{m=1(\neq j)}^{n}\Big[|W_{mm}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|\left|W_j^i(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})\right| - |W_{jm}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|\left|W_m^i(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})\right|\Big]\Bigg\}. \tag{56}$$

Inserting this into (53) and taking into account the relation

$$|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|' = \sum_{m=1}^{n}|W_{mm}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})| \tag{57}$$

which is a direct consequence of the structure of the matrices $W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})$ and $W_{mm}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})$ we obtain

$$y'_{j,i} = \frac{1}{|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|}\left[\Gamma_{ji} - \sum_{m=1}^{n}|W_{j,m}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|y_{m,i}\right]. \tag{58}$$

By the same means for the vector Θ_E (45) we can get a similar relation.

Thus, the determinant of the matrix (52) can be written as a sum of the two other determinants one of which has the last row as a linear combination of other rows and, hence, this determinant vanishes. The matrix of another determinant consists of minors embordering the block $W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})$ in the determinant (28) (up to a common factor). Applying once again the Sylvester identity one finally obtains

$$\left|Y_{NE}^j\right| = \frac{|W_{jE}(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|}{|W(\mathcal{U}_1, \ldots, \mathcal{U}_{N-1})|} \tag{59}$$

which together with (50), (51) and (49) proves the theorem. □

3.4. Transformation of potential

According to (17), to find the potential resulting from a chain of N Darboux transformations we have to resolve the recursion defined in (18) and (19). This is done by the following:

Theorem 2. *Let the matrix F_N be defined by the recursion $F_N = F_{N-1} + Y'_N Y_N^{-1}$, $N = 1, 2, \ldots$ with the initial condition $F_0 = 0$ and $Y_N = L_{(N-1)\leftarrow 0}\mathcal{U}_N$ with the operator $L_{(N-1)\leftarrow 0} = L_{(N-1)\leftarrow(N-2)} \cdot \ldots \cdot L_{2\leftarrow 1} \cdot L_{1\leftarrow 0}$ defined in theorem 1. Then the elements $f_{i,j}^N$ of the matrix F_N are expressed in terms of transformation functions $\mathcal{U}_k$, $k = 1, \ldots, N$ as follows,*

$$f_{i,j}^N = \frac{|W_{ij}(\mathcal{U}_1, \ldots, \mathcal{U}_N)|}{|W(\mathcal{U}_1, \ldots, \mathcal{U}_N)|} \tag{60}$$

where $W(\mathcal{U}_1, \ldots, \mathcal{U}_N)$ is defined in (25) and $W_{ij}(\mathcal{U}_1, \ldots, \mathcal{U}_N)$ is given by (30).

Proof. This theorem is also proved by the perfect induction method. For $N = 1$ one has $Y_1 = \mathcal{U}_1$ and $W(\mathcal{U}_1) = \mathcal{U}_1$. Equation (60) follows from inverting the corresponding matrix.

Suppose that the theorem holds for $N-1$ transformations meaning that the matrix F_{N-1} has the entries

$$f_{i,j}^{N-1} = \frac{|W_{ij}(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|}{|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|}. \tag{61}$$

To prove the statement we have to calculate the value $\tilde{F}_N = Y_N' Y_N^{-1}$. The matrix Y_N' has the derivatives (58) as its entries. Now since

$$\left(Y_N^{-1}\right)_{i,j} = \frac{1}{|Y_N|} A_{ji} \tag{62}$$

where A_{ij} is the cofactor of the element $(Y_N)_{i,j}$ in the matrix Y_N, then for the entries of the matrix $\tilde{F}_N$ one has

$$\tilde{f}_{i,j}^N = \frac{1}{|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|} \frac{1}{|Y_N|} \sum_{l=1}^n \left[\Gamma_{il} A_{jl} - \sum_{m=1}^n |W_{im}(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})| y_{m,l} A_{jl} \right]. \tag{63}$$

To calculate the first term in the square brackets in (63) we use the equation

$$\sum_{l=1}^n \Gamma_{il} A_{jl} = \left|Y_N^{ij}\right| |W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})| \tag{64}$$

where Y_N^{ij} is the matrix obtained from Y_N by replacing in its jth row the determinants $\left|W_j^l(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})\right|$ (see equation (48)) with Γ_{il}, $l = 1,\ldots,n$ defined in the proof of the previous theorem, which follows directly from the decomposition of the determinant $\left|Y_N^{ij}\right|$ on its jth row containing $\Gamma_{il}/|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|$. Another relation to be used here is the product of a matrix with its inverse written in terms of matrix elements

$$\sum_{k=1}^n y_{ik} A_{jk} = \delta_{ij}. \tag{65}$$

Now we rewrite equation (63) as follows:

$$\tilde{f}_{ij}^N = \frac{\left|Y_N^{ij}\right|}{|Y_N|} - \frac{|W_{ij}(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|}{|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|}. \tag{66}$$

According to (61) the last term in (66) represents the elements of the matrix F_{N-1}. As a result for the entries of the matrix F_N one gets

$$f_{i,j}^N = \frac{\left|Y_N^{ij}\right|}{|Y_N|}. \tag{67}$$

The final comment is that the determinants representing numerators of the elements of the matrix Y_N^{ij} (we recall that it coincides with the matrix Y_N composed of the elements (48) except for the jth row composed of the elements $\Gamma_{il}/|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|$) are up to the factor $1/|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|$ the minors embordering the block $W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})$ in the matrix $W_{ij}(\mathcal{U}_1,\ldots,\mathcal{U}_N)$ and the application of the Sylvester identity yields

$$\left|Y_N^{ij}\right| = \frac{|W_{ij}(\mathcal{U}_1,\ldots,\mathcal{U}_N)|}{|W(\mathcal{U}_1,\ldots,\mathcal{U}_{N-1})|}. \tag{68}$$

This equation together with (67) and the expression (49) for $|Y_N|$ proves the theorem. □

4. Conclusion

As a concluding remark we would like to mention that the field of application of the theorems we have just proved is not restricted only by the matrix Schrödinger equation. In their proof we have never used the fact that the transformation functions $\mathcal{U}_k$ are solutions to an equation. Actually, we obtained a solution to a special recursion scheme and, hence, our results may be useful in solving problems where such a scheme appears. For instance, while applying the Darboux algorithm to the stationary Dirac equation we have obtained similar recurrence relations [12].

Another possible application of the above results is related to the realization of phase equivalent chains of transformations for the inverse spectral problem on a semiaxis. Recently conditions on transformation functions have been established for chains of one channel Darboux transformations leading to different phase equivalent potentials [8]. In this way one was able to correct undesirable oscillation of the tail of a shallow potential which was previously obtained in [13] in the frame of the usual supersymmetry approach. The method provides us also with the possibility to get the correct phase shift effective range expansion for radial problems with higher angular momenta (see the second reference of [8]). We hope that the theorems we have established here open the way for working with matrix equations (i.e. Schrödinger or Dirac) almost in such an easy way as is done for the scalar case, and similar investigations are now possible for the multichannel problem.

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Appendix

Here we formulate the Sylvester identity [11]. Consider a square matrix of dimension $p+q,\ p,q=1,2,\ldots$

$$A=\begin{pmatrix} a_{1,1} & \cdots & a_{1,p} & a_{1,p+1} & \cdots & a_{1,p+q} \\ & \cdots & & & \cdots & \\ a_{p,1} & \cdots & a_{p,p} & a_{p,p+1} & \cdots & a_{p,p+q} \\ b_{1,1} & \cdots & b_{1,p} & b_{1,p+1} & \cdots & b_{1,p+q} \\ & \cdots & & & \cdots & \\ b_{q,1} & \cdots & b_{q,p} & b_{q,p+1} & \cdots & b_{q,p+q} \end{pmatrix}. \tag{A1}$$

Let a be the submatrix of dimension $p\times p$ composed of the elements $a_{i,j},\ i,j=1,\ldots,p$. If to the bottom of a we add a line of elements $b_{k,1},\ldots,b_{k,p}$, to the right of a we add a column of elements $a_{1,p+l},\ldots,a_{p,p+l}$ and the right bottom corner we fill with the element $b_{k,p+l}$, we obtain a square matrix $m_{j,l}$. One says that $m_{j,l}$ is obtained from A by embordering the block a with kth row and $(p+l)$th column. The determinant $|m_{j,l}|$ is called an embordering minor in the determinant $|A|$. Since k and l can take the values $k,l=1,\ldots,q$ one has $q\times q$ embordering minors from which one can construct the matrix $M=(m_{j,l})$. The Sylvester identity relates the determinants $|M|$, $|A|$ and $|a|$ as follows:

$$|M|=|a|^{q-1}|A|. \tag{A2}$$

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